Int. J. Solids Structures Vol. 19, No. 11, pp. 1019–1023, 1983
Printed in Great Britain.

0020-7683/83 \$3.00+.00
Pergamon Press Ltd.

UPPER AND LOWER BOUNDS FOR THE ENERGY-RELEASE RATES IN AN ELASTIC BODY

PAWEL RAFALSKI

Institute of Nuclear Research, Swierk near Warsaw, Poland, Visiting Scholar at Stanford University, Stanford, CA 94305, U.S.A.

and

ALICIA GOLEBIEWSKA-HERRMANN

Division of Applied Mechanics, Department of Mechanical Engineering, Stanford University, Stanford, CA 94305, U.S.A.

(*Received* 23 *July* 1982)

Abstract—A numerical method of evaluation of the energy-release increment resulting from the crack propagation in the elastic body is proposed. The method is based on the extremum principles established for the elastic-perfectly plastic body. An example of calculation of the upper and lower bounds for the energy released in edge-crack specimen is given.

1. INTRODUCTION

The problem of bounds in numerical calculations of the energy-release rate is of great interest in fracture mechanics. Bui [1] has indicated that the standard variational approach to the particular energy-release problem for the elastic body does not give satisfactory results. Namely, the bounds obtained for the potential energies before and after the considered crack propagation are not of great value to calculate the bounds for the energy-release rate.

In the present work we shall evaluate the energy-release increment for nonhomogeneous elastic body using the extremum principles established by Rafalski [2] for the elastic-perfectly plastic body. In particular we shall calculate upper and lower bounds for the energy-release resulting from uniform tension of rectangular specimen with a crack.

2. FORMULATION OF THE PROBLEM

We consider 3-dimensional region V bounded by sufficiently regular surface B. The boundary B is decomposed into the surface B_K, where the displacements $\mathbf{u}^B(\mathbf{x})$ are prescribed and the surface B_S, where the tractions $\mathbf{T}^B(\mathbf{x})$ are prescribed. The prescribed initial and final material properties of the body occupying the region V will be referred to as body 0 and body 1, respectively.

The *energy-release increment* resulting from the transition from body 0 to body 1 is defined by

$$\Delta E = \int_V [W_1(\boldsymbol{\epsilon}^1, \mathbf{x}) - W_0(\boldsymbol{\epsilon}^0, \mathbf{x})]\, dV - \int_{B_S} \mathbf{T}^B \cdot (\mathbf{u}^1 - \mathbf{u}^0)\, dB \tag{1}$$

and expresses the difference between the increment of energy stored in the material and the increment of energy supplied from outside. Here $W_0(\boldsymbol{\epsilon}, \mathbf{x})$ and $W_1(\boldsymbol{\epsilon}, \mathbf{x})$ are prescribed *free energy functions* for body 0 and body 1, respectively, defined for all strain tensors $\boldsymbol{\epsilon}$ and all $\mathbf{x}$ from V. It is assumed that these functions are convex and differentiable with respect to $\boldsymbol{\epsilon}$ and that they attain their minima at $\boldsymbol{\epsilon} = \mathbf{0}$. The functions $\boldsymbol{\epsilon}^0(\mathbf{x})$ and $\boldsymbol{\epsilon}^1(\mathbf{x})$ represent the actual strain functions in body 0 and body 1, respectively, and $\mathbf{u}^0(\mathbf{x})$, $\mathbf{u}^1(\mathbf{x})$ are corresponding displacement functions.

The strain and stress functions introduced above satisfy the *constitutive relations*

$$\boldsymbol{\sigma}^0(\mathbf{x}) = \frac{\partial W_0(\boldsymbol{\epsilon}, \mathbf{x})}{\partial \boldsymbol{\epsilon}}\bigg|_{\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^0(\mathbf{x})}, \quad \boldsymbol{\sigma}^1(\mathbf{x}) = \frac{\partial W_1(\boldsymbol{\epsilon}, \mathbf{x})}{\partial \boldsymbol{\epsilon}}\bigg|_{\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^1(\mathbf{x})} \quad \text{in } V \tag{2}$$

the *kinematic conditions*

$$\epsilon^0_{ij} = \frac{1}{2}(u^0_{i,j} + u^0_{j,i}), \quad \epsilon^1_{ij} = \frac{1}{2}(u^1_{i,j} + u^1_{j,i}) \quad \text{in } V \tag{3}$$

$$u^0_i = u^B_i, \quad u^1_i = u^B_i \qquad \text{on } B_K \tag{4}$$

and the *static conditions*

$$\sigma^0_{ij,j} = 0, \quad \sigma^1_{ij,j} = 0 \qquad \text{in } V \tag{5}$$

$$\sigma^0_{ij} n_j = T^B_i, \quad \sigma^1_{ij} n_j = T^B_i \qquad \text{on } B_S \tag{6}$$

where $\mathbf{n}$ is the unit vector normal to the boundary B and taken as positive outwardly.

3. UPPER AND LOWER BOUNDS

To calculate the bounds for the energy-release increment we introduce a family K of *kinematically admissible* strain functions $\tilde{\boldsymbol{\epsilon}}(\mathbf{x})$ defined in the region V, i.e. the functions which can be derived from the displacement function $\tilde{\mathbf{u}}(\mathbf{x})$ $\{\tilde{\epsilon}_{ij} = 1/2(\tilde{u}_{i,j} + \tilde{u}_{j,i})$ in $V\}$ and satisfy the boundary condition $\tilde{u}_i = u^B_i$ on B_K. We also introduce a family S of *statically admissible* stress functions $\tilde{\boldsymbol{\sigma}}(\mathbf{x})$, i.e. the functions which satisfy the equilibrium equation $\tilde{\sigma}_{ij,j} = 0$ in V and the boundary condition $\tilde{\sigma}_{ij} n_j = T^B_i$ on B_S.

It follows directly from the extremum principles [2] established for the elastic-perfectly plastic body that the *upper bound* β_u and the *lower bound* β_l of the energy increment ΔE for the considered elastic problem are expressed by

$$\beta_u = \inf\left\{\int_V [W_1(\tilde{\boldsymbol{\epsilon}}, \mathbf{x}) - \tilde{\boldsymbol{\epsilon}} \cdot \tilde{\boldsymbol{\sigma}} + W^*_0(\tilde{\boldsymbol{\sigma}}, \mathbf{x})]\, \mathrm{d}V : \tilde{\boldsymbol{\epsilon}} \in K, \tilde{\boldsymbol{\sigma}} \in S\right\} \tag{7}$$

$$\beta_l = \sup\left\{\int_V [W_0(\tilde{\boldsymbol{\epsilon}}, \mathbf{x}) - \tilde{\boldsymbol{\epsilon}} \cdot \tilde{\boldsymbol{\sigma}} + W^*_1(\tilde{\boldsymbol{\sigma}}, \mathbf{x})]\, \mathrm{d}V : \tilde{\boldsymbol{\epsilon}} \in K, \tilde{\boldsymbol{\sigma}} \in S\right\} \tag{8}$$

where $W^*_0(\boldsymbol{\sigma}, \mathbf{x})$, and $W^*_1(\boldsymbol{\sigma}, \mathbf{x})$ denote the functions polar (see [2]) to $W_0(\boldsymbol{\epsilon}, \mathbf{x})$ and $W_1(\boldsymbol{\epsilon}, \mathbf{x})$, respectively.

4. APPLICATION TO CRACK PROPAGATION

The evaluation of the energy-release increment presented above can be applied to the crack which propagates in non-homogeneous elastic body. The technique of calculation will be presented with the simple example of edge-crack specimen in tension by uniform displacement (compare with the example in [1]). The rectangular region $V(0 \leqq x_1 \leqq b, -l \leqq x_2 \leqq l)$ of unit thickness containing two triangular subregions V_0 and V_1 $(V_0 \subset V_1 \subset V)$ (see Fig. 1) is extended uniformly (the vertical component of the displacement $\mathbf{u}^B$ is equal to e_0). Body 0 is characterized by sectionally constant free energy function

$$W_0(\boldsymbol{\epsilon}, \mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in V_0 \\ \dfrac{1}{2}\dfrac{E}{1+\nu}\left(\epsilon_{ij}\epsilon_{ij} + \dfrac{\nu}{1-2\nu}\epsilon_{ii}\epsilon_{jj}\right) & \text{otherwise} \end{cases} \tag{9}$$

where E is the Young modulus and ν is the Poisson's ratio. Hence the region V_0, determined by the dimensions a_0 and h, represents the initial shape of the crack. The final shape of the crack is here represented by region V_1, determined by the dimensions a_1 and h. Consequently the free energy function for body 1 is defined by

$$W_1(\boldsymbol{\epsilon}, \mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in V_1 \\ \dfrac{1}{2}\dfrac{E}{1+\nu}\left(\epsilon_{ij}\epsilon_{ij} + \dfrac{\nu}{1-2\nu}\epsilon_{ii}\epsilon_{jj}\right) & \text{otherwise.} \end{cases} \tag{10}$$

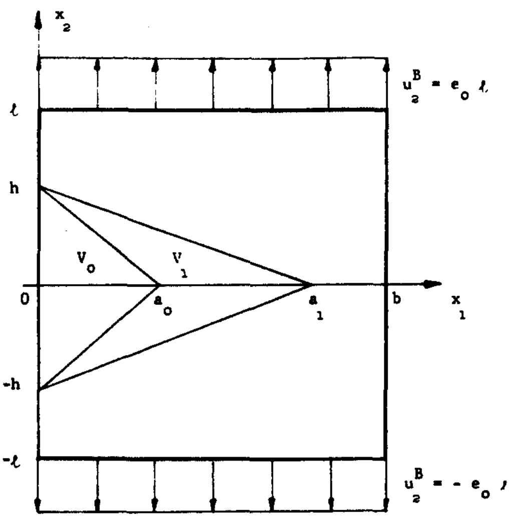

Fig. 1. Edge-crack specimen in tension by uniform displacement.

To calculate the bounds we define two families K_m, $m = 0, 1$ of kinematically admissible strain functions $\bar{\epsilon}(\mathbf{x})$ characterized by free parameter e which determines the displacement $w = he_0 + (l - h)e$ of the point [0, h] (see Fig. 2). It is assumed that the deformation of the region V is sectionally homogeneous, i.e. that every triangular subregion V_m, V_m', V_m'' as well as the remaining part of V are subjected to homogeneous deformation and that $\bar{\epsilon}_{11} = \bar{\epsilon}_{33} = -\nu\bar{\epsilon}_{22}$, $\bar{\epsilon}_{13} = \bar{\epsilon}_{23} = 0$.

We also define two families S_m, $m = 0, 1$ of statically admissible stress functions $\bar{\sigma}(\mathbf{x})$ characterized by free parameter p

$$\bar{\sigma}_{22}(\mathbf{x}) = \begin{cases} 0 & \text{if } x_1 < a_m \\ p & \text{otherwise} \end{cases} \tag{11}$$

$$\bar{\sigma}_{ij}(\mathbf{x}) = 0 \text{ in } V \text{ excluding } i = 2,\ j = 2.$$

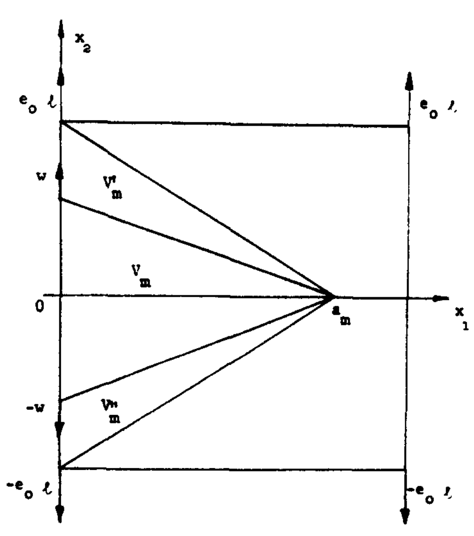

Fig. 2. Kinematically admissible deformation.

Making use of the fact that the specimen is symmetric with respect to axis x_2 we obtain for $\bar{\epsilon}(\mathbf{x}) \in K_m$ and $\bar{\sigma}(\mathbf{x}) \in S_n$

$$\int_V W_m(\bar{\epsilon}, \mathbf{x})\,\mathrm{d}V = \frac{1}{2}\,blE\{(2-\alpha_m)e_0^2 + \alpha_m(1-\eta)[e_0 - e)^2 + \lambda_m e^2]\} \tag{12}$$

$$\int_V \bar{\epsilon}\bar{\sigma}\,\mathrm{d}V = 2blE(1-\alpha_n)pe_0 \tag{13}$$

$$\int_V W_n(\sigma, x)\,\mathrm{d}V = blE^{-1}(1-\alpha_n)p^2 \tag{14}$$

where

$$\eta = \frac{h}{l}, \quad \alpha_m = \frac{a_m}{b}$$

and

$$\lambda_m = \frac{1}{2}\frac{1}{1+\nu}\frac{l^2}{a_m^2}.$$

Substituting (12)–(14) into (7) we obtain the upper bound for the energy release increment corresponding to the crack tip propagation from $[0, a_0]$ to $[0, a_1]$

$$\beta_u = \frac{1}{2}\,blEe_0^2\left\{2\alpha_0 - \left[1-(1-\eta)\frac{\lambda_1}{1+\lambda_1}\right]\alpha_1\right\}. \tag{15}$$

The upper bound is attained in the spaces K_1 and S_0 at $p = Ee_0$ and $e = e_0(1/1+\lambda_1)$. Similarly we obtain the lower bound substituting (12)–(14) into (8)

$$\beta_l = \frac{1}{2}\,blEe_0^2\left\{\left[1-(1-\eta)\frac{\lambda_0}{1+\lambda_0}\right]\alpha_0 - 2\alpha_1\right\}. \tag{16}$$

The lower bound is attained in the spaces K_0 and S_1 at $p = Ee_0$ and

$$e = e_0\frac{1}{1+\lambda_0}.$$

In the particular case of the plane crack ($\eta = 0$) we obtain the inequality

$$\frac{1}{2}\,blEe_0^2\left(\frac{\alpha_0}{1+\lambda_0} - 2\alpha_1\right) \leqq \Delta E \leqq \frac{1}{2}\,blEe_0^2\left(2\alpha_0 - \frac{\alpha_1}{1+\lambda_1}\right). \tag{17}$$

5. CONCLUDING REMARKS

It should be noted that the spaces K_m and S_n introduced in Section 4 are extremely poor as each of them contains only one free parameter. Consequently the result (17) should be considered as a rough approximation. Indeed, assuming that body 0 has no crack ($a_0 = 0$), we obtain for arbitrary a_1

$$\frac{\beta_u - \beta_l}{|\beta_l|} = \frac{1+2\lambda_1}{2+2\lambda_1} > 0.5. \tag{18}$$

It is obvious that more precise evaluation of the energy increment requires more complex spaces of kinematically and statically admissible functions which would provide better approximation of the strain and stress distributions in the specimen. To construct the space of kinematically admissible strain functions the standard finite-element technique can be directly used.

Acknowledgements—This research was performed at the Division of Applied Mechanics of Stanford University. The authors are most grateful to Prof. G. Herrmann of Stanford University for valuable discussions with regard to the context of this paper. This research was supported in part by Air Force Office of Scientific Research Grant AF80-0149 to Stanford University.

REFERENCES

1. H. D. Bui, Dual path independent integrals in the boundary-value problems of cracks. *Engng Fracture Mech.* **6**, 287–296 (1974).
2. P. Rafalski, Extremum principles for the energy-release problem of elastic-perfectly plastic body subjected to prescribed change of material properties. *Int. J. Solids Structures*, submitted for publication.